Test 3 / Numerical Mathematics 1 / June 20th 2023, University of Groningen

Instructions

- **Use a ball-point pen (blue or black ink) to write your answers.**
- You have 2 hours to complete the test. When applicable, people with special facilities have 2h20 minutes in total.
- The exam is "closed book", meaning that you can only make use of the material given to you.
- A simple calculator is allowed.
- The grade will be computed as the number of obtained points, plus 1.
- Not complying with the aforementioned rules will lead to zero points.

Exercises

Consider the system of ODEs for the functions $d(t), v(t)$:

$$\begin{cases} mv' = -k\exp(d)d - cv & v(0) = v_0 \\ d' = v - \gamma d, & d(0) = d_0 \end{cases}, \tag{1}$$

with $m, k. c, \gamma > 0$.

(a) [2] Show that there exists some function $g(d) > 0$ such that:

$$(v^2)' + g(d)(d^2)' \leq 0$$

(b) [1] Discretize the system of equations (1) in time using the β-method, and denote the discrete solutions by $d_n \approx d(t_n), v_n \approx v(t_n)$. Formulate the system of (possibly non-linear) equations for (d_{n+1}, v_{n+1}) as a vector root finding problem $T(d_{n+1}, v_{n+1}) = 0$, and give the specific form for T. Assume $t_{n+1} = t_n + h$, $h > 0$.

(c) [3] Write down the Newton iteration for computing the $(k+1)$-st iterand $d_{n+1}^{k+1}, v_{n+1}^{k+1}$ from the k-th iterand d_{n+1}^k, v_{n+1}^k. Give specific expressions for the vectors and matrix involved, but you do not need to explicitly invert any matrix.

(d) [3] Consider now the system of ODEs:

$$X'(t) = AX(t), \quad X(0) = X_0 \neq 0, \tag{2}$$

with

$$A = \begin{bmatrix} -1 & -1/2 \\ 1/2 & -2 \end{bmatrix}.$$

Discretize the ODE (2) using the β-method for $\beta = 0$. Then, determine h_{crit} such that $X_n \to 0$ (for any X_0) when $n \to \infty$ for $h < h_{crit}$.